THE FIFTEENTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

SINCE the founding of the Society in 1888, the regular, including the annual, meetings have been held almost without exception in New York City, as the most convenient center for the members living in the eastern states and others who might from time to time attend an eastern meeting. The summer meeting, migratory between limits as far apart as Boston and St. Louis, has afforded an annual opportunity for a fully representative gathering, and provision has been made for the convenience of the central and western members by the founding of the Chicago Section in 1897, the San Francisco Section in 1902, and the Southwestern Section in 1906. The desire has, however, often been expressed that the annual meeting of the Society might, when geographic and other conditions were exceptionally favorable, be occasionally held like that of many other scientific bodies in connection with the meeting of the American association for the advancement of science, a gathering which naturally affords many conveniences of travel and scientific advantages. It was therefore decided to hold the annual meeting of 1908 at Baltimore in affiliation with the Association, the days chosen being Wednesday and Thursday, December 30–31.

Two sessions were held on each day in the Biological Laboratory of Johns Hopkins University. The total attendance numbered about seventy-five, including the following fifty-seven members of the Society:

Miss C. C. Barnum, Dr. E. G. Bill, Professor G. A. Bliss, Professor E. W. Brown, Dr. A. B. Chace, Professor A. S. Chessin, Dr. A. B. Coble, Dr. Abraham Cohen, Professor F. N. Cole, Dr. G. M. Conwell, Miss E. B. Cowley, Mr. F. F. Decker, Mr. C. E. Dimick, Professor John Eiesland, Professor L. P. Eisenhart, Professor J. C. Fields, Dr. Fabian Franklin, Dr. W. A. Granville, Dr. F. L. Griffin, Professor C. C. Grove, Professor Harris Hancock, Professor J. G. Hardy, Professor C. N. Haskins, Rev. A. S. Hawkesworth, Dr. A. A. Himowich, Professor H. L. Hodgkins, Professor L. S. Hulbert, Professor J. I. Hutchinson, Professor Edward Kasner, Profes-

sor C. J. Keyser, Mr. W. J. King, Mr. Joseph Lipke, Dr. W. R. Longley, Professor W. H. Maltbie, Professor Mansfield Merriman, Professor J. F. Messick, Professor G. A. Miller, Professor J. A. Miller, Dr. C. L. E. Moore, Professor Frank Morley, Professor G. D. Olds, Professor N. A. Patillo, Dr. H. B. Phillips, Professor E. D. Roe, Professor F. H. Safford, Miss M. E. Sinclair, Professor P. F. Smith, Professor Virgil Snyder, Professor H. F. Stecker, Professor J. H. Tanner, Professor E. J. Townsend, Professor H. W. Tyler, Professor Anna L. Van Benschoten, Professor C. A. Waldo, Professor H. S. White, Professor T. W. D. Worthen, Professor J. E. Wright.

Wednesday evening was set apart for a social reunion and dinner, at which forty-five members were present. The meeting of the Council was held later on the same evening.

The annual meeting was especially marked as the occasion of the retiring address of President H. S. White on "Bezout's theory of resultants and its influence on geometry," which was delivered at the opening of the afternoon session on Wednesday.

The heavy programme fully occupied the four sessions. President White occupied the chair, being relieved by Professor Morley and Vice-Presidents G. A. Miller and Kasner. The Council announced the election of the following persons to membership in the Society: Professor G. N. Armstrong, Ohio Wesleyan University; Professor P. F. Gaehr, Robert College, Constantinople; Dr. Frank Irwin, Princeton University; Miss Mary E. Wells, Mount Holyoke College. Six applications for membership were received.

The reports of the Treasurer, Auditing Committee, and Librarian will appear in the Annual Register, now in press. The membership of the Society has increased during the past year from 582 to 601, including at present 55 life members. The number of papers presented at all meetings during the year was 169. The total attendance of members at the meetings was 281. The Treasurer's report shows a balance of $6995.80, of which $3215.20 is credited to the life membership fund. Sales of the Society's publications during the year amounted to $1385.36.

At the annual election, which closed on Thursday morning, the following officers and members of the Council were chosen:

President,	Professor MAXIME BÔCHER.
Vice-Presidents,	Professor EDWARD KASNER.
	Professor E. B. VAN VLECK.

Secretary,	Professor F. N. COLE.
Treasurer,	Professor J. H. TANNER.
Librarian,	Professor D. E. SMITH.

Committee of Publication,
Professor F. N. COLE,
Professor D. E. SMITH,
Professor VIRGIL SNYDER.

Members of the Council to serve until December, 1911,

Professor H. B. FINE, Professor F. R. MOULTON,
Professor O. D. KELLOGG, Professor E. J. WILCZYNSKI.

The following papers were read at this meeting:

(1) Professor R. D. CARMICHAEL: "On r-fold symmetry of plane algebraic curves."

(2) Professor R. D. CARMICHAEL: "A general principle of inversion, with applications."

(3) Dr. W. R. LONGLEY: "Some sufficient conditions in the theory of implicit functions."

(4) Dr. C. L. E. MOORE: "Properties of systems of lines in space of four dimensions and their interpretation in circle geometry."

(5) Dr. F. R. SHARPE: "The topography of the integral curves of a differential equation."

(6) Mr. JOSEPH LIPKE: "Note on isotropic ruled surfaces."

(7) Professor JOHN EIESLAND: "On a species of cubic surfaces of the sixth class."

(8) President E. O. LOVETT: "Integrable problems of three bodies."

(9) Professor J. I. HUTCHINSON: "On linear transformations which leave an hermitian form invariant."

(10) President H. S. WHITE: — Presidential Address: "Bezout's theory of resultants and its influence on geometry."

(11) Professor FRANK MORLEY: "Plane sections of the Weddle surface."

(12) Professor G. A. MILLER: "Finite groups which may be defined by two operators satisfying two conditions."

(13) Dr. F. L. GRIFFIN: "Tests comparing the apsidal angles and periodic times for different laws of central force."

(14) Dr. E. G. BILL: "Existence 'im Kleinen' of a space curve which minimizes a definite integral."

(15) Dr. E. G. BILL: "An a priori existence theorem in three dimensions for the calculus of variations."

(16) Mr. J. R. CONNER: "Curves and surfaces which admit configurations of the Cayley-Veronese type."

(17) Mr. D. D. LEIB: "The complete system of invariants for two triangles."

(18) Dr. W. A. GRANVILLE: "Dual formulas in spherical trigonometry."

(19) Professor C. J. KEYSER: "Concerning euclidean geometries without points and lines."

(20) Miss M. E. SINCLAIR: "The problem of the surface of revolution with two end points variable on circles."

(21) Professor G. A. BLISS: "On the construction of the coordinate system of analytic projective geometry."

(22) Professor EDWARD KASNER: "The group generated by turns and slides."

(23) Professor EDWARD KASNER: "Catenaries in an arbitrary field of force."

(24) Professor C. N. HASKINS: "Numerical computation of reaction velocity constants."

(25) Dr. W. B. CARVER: "Degenerate pencils of quadrics connected with $\Gamma^{n+2}_{n+4,\,n}$ configurations."

(26) Dr. C. F. CRAIG: "On a class of hyperfuchsian functions."

(27) Professor VIRGIL SNYDER: "Surfaces and congruences derived from the cubic variety having a double line in four-dimensional space."

(28) Professor W. B. FITE: "Irreducible homogeneous linear groups in an arbitrary infinite field."

(29) Dr. ARTHUR RANUM: "On certain solids in riemannian space."

(30) Dr. ARTHUR RANUM: "On the rank of a matrix."

(31) Dr. H. B. PHILLIPS: "Polygons on a quadric surface."

(32) Professor J. G. HARDY: "Note on a theorem of Pirondini concerning four-dimensional curves."

(33) Dr. A. B. COBLE: "Combinants of binary forms."

Mr. Conner and Mr. Leib were introduced by Professor Morley. In the absence of the authors the papers of Dr. Sharpe, Dr. Carver, Dr. Craig, Professor Fite, and Dr. Ranum were read by Professor Snyder, and the papers of Professor Carmichael, President Lovett, and Dr. Phillips were read by title. President White's address will appear in the next num-

ber of the BULLETIN. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. This paper is a generalization of a preceding one on fourfold symmetry of plane algebraic curves. Professor Carmichael shows that there are still two classes of curves distinguished by differing forms of the equation. Taking the general cartesian equation in the form

$$\sum a_{ts}x^t y^s = 0 \qquad (t+s \leqq n),$$

with origin at the center of symmetry, it is shown that for r-fold symmetry the necessary and sufficient relations among the constants a in the two classes are respectively

$$\sum_{t,\,s=0}^{t,\,s=v} a_{ts}\cos^t\theta_1 \sin^s\theta_1 = \sum_{t,\,s=0}^{t,\,s=v} a_{ts}\cos^t(\theta_1+\alpha\phi)\sin^s(\theta_1+\alpha\phi), \qquad (t+s=v),$$

$$\sum_{t,\,s=0}^{t,\,s=v} a_{ts}\cos^t\theta_1 \sin^s\theta_1 = (-1)^{\alpha}\sum_{t,\,s=0}^{t,\,s=v} a_{ts}\cos^t(\theta_1+\alpha\phi)\sin^s(\theta_1+\alpha\phi), \qquad (t+s\neq v),$$

where $v = 0, 1, 2 \cdots, n$ (different equations being formed for each value of v), α is any integer $\leqq r-1$, and $\phi = 360°/r$. Methods are given for determining the a's in each case and the results are worked out for the smaller values of r and n. Finally, an example is given of a special tenfold symmetrical curve.

2. G. Cantor has developed several properties of the function $\beta(N)$ defined by

$$\beta(1) = 1, \quad \beta(N) = (-1)^{a_1+a_2+\cdots+a_n},$$

where $N = p_1^{a_1}p_2^{a_2}\cdots p_n^{a_n} > 1$, $p_1, p_2, p_3, \cdots, p_n$ being different primes. Professor Carmichael establishes the following inversion principle:

If n is an integer and $f(n)$ and $F(n)$ are two functions connected by the relation $f(n) = \sum_k F(kn)$, but otherwise quite arbitrary, then $\sum_n F(n^2) = \sum_n \beta(n)\cdot f(n)$, where the summation may extend to an infinite number of terms or may stop short at a finite number of terms by the assumption that $f(n)$ and $F(n)$ have only the value zero when n is greater than some fixed number.

Several applications are made of this principle. In particular it is applied to the summation of certain classes of infinite series.

3. In Dr. Longley's paper the functions considered are real functions of real variables. The following theorem on implicit functions is well known: Consider the system of equations

$$(1) \qquad f_i(x_1, \cdots, x_n;\ u_1, \cdots, u_m) = 0 \quad (i = 1, \cdots n).$$

Suppose a special solution

$$(2) \qquad x_1 = a_1, \cdots, x_n = a_n;\ u_1 = b_1, \cdots, u_m = b_m,$$

where a_i and b_i denote constants, is known, and that in the neighborhood of the system of values (2) the functions f_i are of class * C'. Suppose the functional determinant

$$\Delta = \frac{\partial(f_1, \cdots, f_n)}{\partial(x_1, \cdots, x_n)} \neq 0$$

for the system of values (2). Then equations (1) have one and only one solution of the form

$$A: \qquad x_i = \phi_i(u_1, \cdots, u_m),$$

where $a_i = \phi_i(b_1, \cdots, b_m)$.

The hypothesis of this theorem states a condition which is sufficient to insure the existence of a unique solution of the type A. The condition $\Delta \neq 0$ is however not necessary. Supposing that $\Delta = 0$, it is the object of this paper to present some other conditions which are sufficient to insure the existence of a unique solution of the type A. The geometric significance of some of the theorems is considered. Applications are made to the existence, in the plane and in space, of a field of extremals about a point where all the extremals of the set pass through that point.

4. If the equations of a line in a space of four dimensions S_4 be written in the form

$$x_1 = a_1 + \alpha_1 z, \quad x_2 = a_2 + \alpha_2 z, \quad x_3 = a_3 + \alpha_3 z, \quad x_4 = z,$$

* That is, continuous with continuous first partial derivatives.

then the equations of the tangent plane to a ruled surface in S_4 may be written

$$\frac{x_1 - a_1 - \alpha_1 x_4}{x_2 - a_2 - \alpha_2 x_4} = \frac{a_1' + \alpha_1' z}{a_2' + \alpha_2' z} = \rho_1,$$

$$\frac{x_1 - a_1 - \alpha_1 x_4}{x_3 - a_3 - \alpha_3 x_4} = \frac{a_1' + \alpha_1' z}{a_3' + \alpha_3' z} = \rho_2.$$

Considering a_1', a_2', a_3', α_1', α_2', α_3' as the homogeneous coordinates of the projectivity between the points z of any generator and the planes passing through the generator and tangent to the ruled surface, Dr. Moore discussed the general properties of lines in space of four dimensions and interpreted the results in circle geometry in space of three dimensions.

5. Dr. Sharpe's paper discusses the topography of the integral curves of $dy/dx = c_1/c_2$ where $c_1 = 0$, $c_2 = 0$ are conics. The nature of the integral curves depends on the locus of points of the conic $c_1 = c_2\lambda$ at which the conic has the slope λ. This locus is also the locus of the inflections of the integral curves and is a quintic having a double point at each of the four intersections of $c_1 = 0$, $c_2 = 0$ and passing through the three intersections of the three pairs of lines through the four points. When the conics $c_1 = 0$, $c_2 = 0$ have simple contact, the quintic has a triple point at the point of contact. When the contact is of the second order, the triple point consists of a cusp and a simple branch. When the contact is of the third order, the quintic degenerates into the common tangent and a quartic having a tacnode at the point of contact. Examples are given of the most interesting cases.

6. In discussing the problem of the line of striction on a ruled skew surface, it is generally shown that an indeterminacy arises if the generator is a minimal straight line. In this note, Mr. Lipke defines the line of striction in one of the usual ways and, introducing infinitesimals of the second order, gets a definite curve on the isotropic skew surface as its line of striction. He further discusses the relations existing between this curve, the edge of regression of the minimal developable enveloped by the system of minimal planes through the generators, and the focal curve of the congruence of normals to the surface.

7. Professor Eiesland's paper is a study of cubic surfaces of the form

$$A + Bx + Dy + Cz + Exz + Fxy + Gyz + Hxyz = 0,$$

with the identical relation $EGAF = HDCB$ between the coefficients. It is shown that two pairs of families of twisted cubics exist on it such that by the involutory transformation

$$\bar{x} = \frac{\lambda}{x}, \quad \bar{y} = \frac{\mu}{y}, \quad \bar{z} = \frac{\nu}{z},$$

where λ, μ, ν are properly chosen values, each pair is transformed into the other. These surfaces are connected with a unicursal quartic in the plane at infinity.

8. Generalizing a well-known problem of Bertrand in mechanics, Oppenheim showed in 1894* that if three given particles describe under central conservative forces three given plane curves, the forces are derived from a potential function which may be written in a form in which there appear only the constant of areas, the masses of the bodies, their coordinates referred to the center of gravity of the system, and the first partial derivatives of the functions defining the orbits. Oppenheim applied his results to the rediscovery of the known solutions of the plane problem of three bodies in which the law of force is as the masses, and either directly as the distance or inversely as the square of the distance.

In Professor Lovett's paper, by the aid of the theory of partial differential equations of the first order and the form of the force function revealed in Oppenheim's memoir, there is constructed an infinitude of integrable problems of three bodies in the plane under forces depending on the masses and mutual distances of the bodies. These problems are grouped in four families according to the form of the potential function. The paper concludes with a fifth family in which the coordinates of the bodies appear explicitly in the force function.

9. The object of the paper by Professor Hutchinson is to give some of the properties of linear transformations on n variables for which an hermitian form H is invariant. The substitutions whose characteristic determinants have no equal roots are classified into elliptic (all roots unimodular) or hyperbolic of q differential types (when $2q$ of the roots are not unimodular).

* *Publicationen der v. Kuffner'schen Sternwarte*, vol. 3, pp. 71–95.

The fixed points of an elliptic substitution are situated so that p of them are inside and $n-p$ outside the locus $H=0$ (p being the number of positive terms of H and $p \leqq n-p$). The fixed points for a hyperbolic substitution of the qth type include q pairs which are on H, $p-q$ points inside H, and $n-p-q$ outside H ($q \leqq p$). The generalized Poincaré series is considered, and the form of its expansion in the vicinity of a fixed point determined.

11. The plane section of a Weddle surface is not an arbitrary quartic curve, but one for which an invariant vanishes. The curve contains a configuration B^6, namely, where it is cut by the lines on the surface. The theorem proved by Professor Morley is that it contains an infinity of such configurations.

12. It is evident that two or more distinct operators, none of which is identity, cannot be sufficiently restricted by a single condition of the form $s_1^{\alpha}s_2^{\beta}s_1^{\gamma}s_2^{\delta}\cdots = 1$, where $\alpha, \beta, \gamma, \delta, \cdots$ are positive or negative integers, to generate only groups whose orders are limited by the values of $\alpha, \beta, \gamma, \delta, \cdots$. On the other hand it is possible to find pairs of such conditions which are such that each pair is satisfied by two generators of only a limited number of groups. Such a pair cannot be of the form $s_1^{\alpha} = 1$, $s_1^{\beta} = 1$ since the order of the product of two operators is not limited by the orders of these operators. In the present paper Professor Miller considers only those cases where a pair of conditions between two operators s_1, s_2 determines either only a single group or a limited number of groups.

If the two conditions are of the form $s_1^n = 1$, $s_1^{\alpha} = s_2^{\beta}$ the order of G is limited when n and α are relatively prime, and only then. When these conditions are satisfied, G is cyclic and its order divides βn. For instance, the cyclic group of order 40 is completely defined by the two conditions $s_1^{10} = 1$, $s_1^3 = s_2^4$. On the other hand, the two conditions $s_1^{10} = 1$, $s_1^4 = s_2^3$ are satisfied by two generators of an infinite number of distinct groups. The following theorem is useful in the study of some of the pairs of conditions: If two commutative operators satisfy the condition $s_1^n = s_2^m$, they generate the direct product of two cyclic groups whose orders are respectively the lowest common multiple of the orders of these operators, and a divisor of the highest common factor of α, β. The latter group may be identity, and it must be identity whenever α, β are relatively prime.

A large number of interesting results are obtained when the two conditions are of the form $s_1^n = 1$, $s_1 s_2 = s_2^\alpha s_1^\beta$. Among these are the following: These two conditions are satisfied by the two generators of only a finite number of groups when $\beta = 1$ and $\alpha \neq 1$, but they are satisfied by two generators of each one of an infinite system of distinct groups when also $\alpha = 1$. If two generators of a group satisfy the conditions $s_1^3 = 1$, $s_1 s_2 = s_2^2 s_1^{-1}$, the group must be either tetrahedral or the group of order 24 involving no subgroup of order 12.

13. In an earlier paper * Dr. Griffin proved certain theorems concerning the apsidal angle Θ in central orbits, obtaining, for example, conditions upon any given law of force necessary and sufficient in order that $\Theta > \pi$ in all orbits of any region. An examination of the method of proof, which consisted of a comparison of the definite integrals giving Θ and π (the corresponding angle for Newton's law), suggests the possibility of generalizing the theorems so as to compare the values of Θ in orbits under any two arbitrarily chosen laws of force.

Several such generalizations are given in the present paper, and applications are made to various simple laws of force. The new proofs, while more involved than those of the former paper, proceed along the same general lines. The criteria again are very simple and easily applied.

Analogous theorems are established for the periodic times in the orbits.

14. Hilbert in 1899 stated that under certain conditions a solution of the calculus of variations problem always existed, and later sketched a proof of this statement which Professor Bolza has since completed.† In this proof the integrand of

$$\int_{t_0}^{t_1} F(x, y, x', y')dt$$

is defined only for a finite closed region and the minimizing curve is built up as the limiting position of an "infinite" sequence of division points of a set of curves which approximate to giving the integral its minimum value.

In Dr. Bill's papers the similar question is considered for the integral

* Cf. BULLETIN, vol. 14 (Oct., 1907), pp. 6–16.

† Bolza, Lectures on the Calculus of Variations, p. 245.

$$J = \int_{t_0}^{t_1} F(x, y, z, x', y', z')dt,$$

carried over a curve in space.

In his first paper it is shown that a solution "im Kleinen" exists, the proof depending on the existence of a field of extremals through a given point.

15. In his second paper Dr. Bill proves the existence "im Grossen," by a method whereby the minimizing curve is obtained by applying the existence theorem "im Kleinen" to a "finite" number of points which are defined as limiting points of points lying on a sequence of approximating curves. Considerable simplification over the original form of the method of Hilbert is thus obtained.

16. Mr. Conner's paper is in abstract as follows: In a space of four dimensions, lines bisecant to a curve C_p^m, of order m and genus p, lie on a 3-way spread Λ of order $h=\frac{1}{2}(m-2)(m-3)-p$, in which C_p^m is an $(m-2)$-fold curve. A space α meets Λ in a surface K_h^m of order h. Denote by Δ_m and Γ_σ, respectively, the plane section of a complete m-point in three dimensions, and the space section of a complete σ-point in four dimensions. σ points on C_p^m define a configuration Γ_σ inscribed in K_h^m. A plane section of Λ admits Δ_m's determined by the sets of m points in which spaces on the plane meet C_p^m. C_0^4 gives rise to a 4-nodal cubic surface K_3^4. The theory of the K_3^4 is connected with that of the rational space quartic curve, the underlying quartic being here the (class) quartic in which the tangential developable to C_0^4 meets α. The surface K_5^5 is the general quintic surface with five triple points. The surface K_6^5 has besides five triple points a double curve of order 7. Curves C_p^m in five dimensions determine in a similar way space curves admitting Γ_m's.

17. In this paper Mr. Leib derives a complete system of the invariants of two triangles, and proves its completeness. Various relations are discussed, including the condition on two triangles so that a conic may be inscribed in one and circumscribed about the other. This leads at once to the equation of the locus of the foci of parabolas through three points, and properties of this locus are deduced directly from the invariant form. The covariants of special interest are also discussed.

18. As generally given by writers on spherical trigonometry, the formulas for the sides of a spherical triangle in terms of the angles have had a certain resemblance to the formulas giving the angles in terms of the sides, but the correspondence has not been perfect. Möbius was the first to point out the fact that if the supplements of the angles of the triangle were used, instead of the angles themselves, one set of these formulas is the dual of another. He did not, however, publish any such sets of formulas, and in this paper Dr. Granville shows what the ordinary formulas of spherical trigonometry become when given in this form. The following theorem is first established:

In any relation between the parts of a general spherical triangle, each part may be replaced by the supplement of the opposite part, and the relation thus obtained will hold true.

When the above theorem is applied to a relation involving one or more of the sides and the supplements of the angles of a spherical triangle, we get the following principle of duality:

If the sides of a general spherical triangle are denoted by the Roman letters a, b, c, and the supplements of the corresponding opposite angles by the Greek letters α, β, γ, then from any given formula involving any of these six parts, we may write a dual formula by simply interchanging the corresponding Greek and Roman letters. The old and the new sets of formulas are then compared, and spherical triangles are solved using both.

19. A system of postulates equivalent to the system (H) of Hilbert's Foundations or to Veblen's system (V) of geometric axioms (*Transactions*, volume 5)—happily described by the latter as categorical—define, not a class s of objects satisfying the system, but a class or category C of such classes s. C is determined by various systems, and is composed of manifold classes s. To each s corresponds a geometric theory determined by any one of the systems and the same for all of them. These theories, though one logically, are many psychologically. Hence the interest attaching to study directed upon the varied content as distinguished from the definition of C. If e, e', e'' denote the undefined elements of (H), e will represent the undefined element of (V), e' and e'' being in the latter system defined by means of the undefined notion of order. The problem is to find near-lying interpretations of the elements, *i. e.*, to discover such components s of C as are easily accessible and

hence interesting to intuition. After the usual interpretation, the most familiar one is that in which s is ordinary space bereft of a point O, e means point, and e' and e'' denote respectively the circles and spheres that would contain O if it were not excluded. In Professor Keyser's paper the following classes s were indicated: (1) s is ordinary space (of planes), lacking any chosen sheaf or bundle o of planes, e denotes ordinary plane, e' denotes pencil of planes lacking the plane belonging to o, and e'' denotes sheaf of planes lacking the e' belonging to o; (2) s is the same as before, e denotes sphere containing the vertex P of o, e' denotes sphere range (or circle) determined by two e's, and e'' denotes sphere congruence (or circle congruence) determined by three e's not belonging to a same e'. The (H) and (V) axioms for two dimensions are satisfied by the following: (1) s is the ordinary plane of lines lacking any chosen pencil o, e denotes line, and e' denotes pencil lacking the e of o; (2) s is same as before, e is circle through the vertex P of o, and e' is circle range determined by two e's; (3) s is a sheaf of planes minus a pencil o, e is plane, and e' is pencil of planes minus the e belonging to o; (4) s is a sheaf of lines lacking a pencil o, e is line, and e' is pencil of lines minus the e of o. Another s is a sheaf of circular cones of common vertex minus a pencil of them.

20. Miss Sinclair's paper is in abstract as follows: The original Plateau experiment of a catenoid film between two anchor rings is the problem of the surface of revolution of minimum area in case the two end points are variable on two circles equal and equidistant from the axis of revolution. Let the x-axis be the axis of revolution, and the y-axis pass through the center of the first circle. Let ρ be the radius of the circles, d the distance of their centers from the axis, (κ, a) the vertex of the catenary. Then we have the equations

$$x_0 = \rho \cos\theta, \quad y_0 - d = \sin\theta; \tag{1}$$

$$y_0 = aC(u_0), \quad x_0 = au_0 + \kappa; \tag{2}$$

$$1 - \frac{x_0}{y_0 - d} S(u_0) = 0, \tag{3}$$

which are sufficient to determine x_0, y_0, a, κ if u_0 is known. On account of the symmetry of the problem, x_1, y_1 can be found from (2) by replacing u_0 by $-u_0$.

The parameter u_0'' of the point focal to u_0 is the positive solution of

$$\frac{\rho}{a} = C^2(u_0)S(u_0)\frac{\phi(u_0'') - \phi(u_0)}{\phi(u_0'') + u_0 + C(u_0)S(u_0)}, \tag{4}$$

where

$$\phi(u_0) = \frac{C(u_0)}{S(u_0)} - u_0.$$

The value of u_0 is restricted by the conditions for fixed end points, on account of the symmetry of the problem, to the interval $(u_0^*, 0)$ where u_0^* is the negative solution of $\phi(u_0) = 0$. The Bliss condition for two variable end points requires for one problem that $\phi(u_0'') \leqq 0$. For the particular value u_0'' for which $\phi(u_0'') = 0$ we obtain, from the above condition,

$$\frac{\rho}{d} = \frac{-C(u_0)S(u_0)\phi(u_0)}{[1-S(u_0)]u_0 + 2C(u_0)S(u_0)}. \tag{5}$$

Moreover, ρ/d is a continuous function of u_0 in the interval $(u_0^*, 0)$, increasing from 0 to ∞. Hence if ρ and d are given, (5) may on this interval be solved uniquely for u_0. Let this solution be $\bar{u}_0$. If ρ and d are constant, u_0'' is found to be an increasing function of u_0, and the Bliss condition is satisfied for $u_0 > \bar{u}_0$.

A physical illustration is of interest and the results of experiment are here given:

$$\rho = 2.5 \text{ cm.}, \quad \theta = 33°,$$

$$d = 5 \text{ cm.}, \quad \text{length of film} = 2\kappa = 3.08.$$

The corresponding theoretical results are (ρ and d being given)

$$\rho = 2.5 \text{ cm.}, \quad \theta = 33° 0' 30'',$$

$$d = 5 \text{ cm.}, \quad 2\kappa = 3.01.$$

21. In Professor Bliss's paper it is shown how the projective coordinates of points and lines in the plane may be defined in terms of anharmonic ratios, and how the relation

$$u_1x_1 + u_2x_2 + u_3x_3 = 0$$

expressing the incidence of the point (x_1, x_2, x_3) and the line (u_1, u_2, u_3) may be proved directly from the properties of an-

harmonic ratios. The definition of the coordinates is not new, but the proof of the incidence relation seems more direct than those usually given.

22. The transformations considered in Professor Kasner's first paper operate on the oriented lineal elements of the plane. By a turn T_α each element rotates about its own point through a fixed angle α. By a slide S_κ each element moves along its own line through a fixed distance κ. It is shown that every combination of turns and slides is reducible to the form $T_\alpha S_\kappa T_\beta$, so that a continuous group of three parameters is obtained. The only contact transformations included are the dilatations. Applied to a simply infinite system of curves the turns lead to the familiar isogonals and the slides to the equitangentials whose theory has been developed recently by Scheffers. The combination of both constructions leads to ∞^3 related systems (though in special cases, admitting automorphic transformations, there will be fewer). The two known theories of trajectories, which appear isolated in the usual treatment, are shown to be special cases of one general theory. The fundamental figure considered, which includes the point, line, and circle as special cases, is formed of ∞^1 elements whose points are on a circle and whose directions are equally inclined to that circle. It is termed a geometric *turbine*. The most general transformation of turbines involves 15 parameters. The resulting geometry is isomorphic with projective geometry in three dimensions. Many previous theories appear in a new light.

23. The study of the equilibrium of a homogeneous flexible inextensible string in a given field of force leads to ∞^5 curves in space, or ∞^3 in the plane. These general systems of catenaries are explored in Professor Kasner's second paper. The results are analogous to but distinct from those dealing with dynamical trajectories (*Transactions*, 1906, 1907). Characteristic properties are obtained, largely by intrinsic methods. It is shown that the field of force may be constructed when a sufficient (infinite) number of its catenaries are known. The modifications necessary for non-homogeneous strings are considered.

24. The paper of Professor Haskins has reference to a problem of chemical dynamics. The course of an incomplete bimolecular reaction is determined by the differential equation

$$dx/dt = k_1(a_1x^2 + 2b_1x + c_1) - k_2(a_2x^2 + 2b_2x + c_2),$$

where a_1, b_1, c_1, a_2, b_2, c_2 are constants known from the initial conditions, and k_1, k_2 are the so-called "reaction constants" which are to be determined from simultaneous observations of x and t. In the reactions hitherto studied the ratio k_1/k_2 has been found from the equilibrium conditions ($t = \infty$) and in this case the computation of k_1/k_2 offers no difficulty. In recent investigations of gas reactions the equilibrium conditions were not attainable, and a method of determination of k_1, k_2 from any two pairs of values of x and t became necessary. The determination can be reduced to the solution of the equation in γ

$$\log\frac{\gamma+z_2}{\gamma-z_2} - \frac{t_2}{t_1}\log\frac{\gamma+z_1}{\gamma-z_1} + \left(\frac{t_2}{t_1}-1\right)\log\frac{\gamma+z_0}{\gamma-z_0} = 0,$$
$$(z_0 < z_1 < z_2, \quad t_1 < t_2),$$

and in important special cases to that of

$$\log\frac{\gamma+z_2}{\gamma-z_2} - \frac{t_2}{t_1}\log\frac{\gamma+z_1}{\gamma-z_1} = 0, \quad (z_1 < z_2, \quad t_1 < t_2).$$

A single condition must be satisfied in order that a root may exist, and the root if existent is unique. The numerical solution proceeds by iteration and converges in general with considerable rapidity.

25. In a paper presented to the Society at the December meeting, 1906 (which has since appeared in the January number of the *American Journal of Mathematics*), Dr. Carver called attention to certain pencils of quadrics connected with the configuration $\Gamma^{n+2}_{n+4,\,n}$. The purpose of the present note is to investigate the question of the existence of degenerate pencils for configurations which are not degenerate.

26. Alezais (*Annales de l'Ecole normale supérieure*, (3), volume 19 (1902), page 261) has studied a monodromy group of a particular Riemann surface. Dr. Craig's paper studies a simpler group and functions belonging to the group. An extension of a known result on the number of linearly independent moduli belonging to an abelian integral of the first kind on a

binomial Riemann surface, and the upper limit for the number of generators of the maximum monodromy group connected with such surfaces are obtained.

27. By the same methods as were employed in the study of cubic varieties having nine double points (abstract in the BULLETIN, volume 15, page 67) Professor Snyder examines all the forms having one or more double lines, filling in the details outlined by the Segre memoirs and supplying a large number of particular cases. All the particularizations of the (3, 2) congruence having a focal line can be obtained and classified from this standpoint, and also the forms of the (2, 2) congruences, the latter being already known. Finally, a large number o surfaces are obtained which remain invariant under birational transformations of infinite order.

28. The principal result in Professor Fite's paper is the following theorem: A necessary and sufficient condition that any group of finite order be simply isomorphic with an irreducible group in any infinite field is that its central be cyclic (or identity). As a direct application of this theorem, the number of those substitutions of finite order that are commutative with a given substitution that is irreducible in any infinite field is determined.

29. Defining a parallelogram in riemannian space as a portion of a Clifford surface bounded by a skew quadrilateral whose opposite sides are right parallels and left parallels, respectively, and a parallelepiped as a solid bounded by three pairs of opposite faces, all of which are parallelograms, Dr. Ranum finds that there are two distinct species of parallelepipeds, one depending on six parameters and the other on two parameters. Defining a quasi-parallelogram as a plane quadrilateral whose opposite sides are congruent, and a quasi-parallelepiped in a corresponding manner, he shows that while there exist oblique quasi-parallelograms (having no lines of symmetry), nevertheless oblique quasi-parallelepipeds (having no planes of symmetry) do not exist.

30. In order to determine the rank of a matrix it is sometimes of practical utility to know the minimum number of determinants of a given order which must be examined as to their vanishing or non-vanishing. In this paper Dr. Ranum con-

siders a matrix having m rows and n columns, and under certain hypotheses as to the non-vanishing of one or more r-rowed determinants of the matrix finds the minimum number of s-rowed determinants ($s > r$) whose vanishing will insure the vanishing of all the other s-rowed determinants.

31. The polygons of Dr. Phillips's paper are formed of groups of $2n$ points joined in order by generators of a quadric surface. From a remark of Salmon it follows that if each odd point is joined to the non-adjacent even points the $n(n-2)$ lines lie on a surface of class $n-2$. The object of the paper is a discussion of these surfaces.

32. Professor Hardy's paper takes up a theorem of Pirondini concerning "cylindrical helices" in a space of four dimensions and shows that under Pirondini's assumptions such curves lie entirely in a properly chosen three dimensional space, and are not, properly speaking, four-dimensional curves.

33. The involution $I^n_{\kappa, n-\kappa}$ of ∞^κ sets of n values such that κ values in a set determine the remaining $n-\kappa$ is expressed analytically by a *special* symmetric binary form in $\kappa+1$ variables to the order $n-\kappa$,

$$I_{\kappa, n-\kappa} \equiv (a_1 x_1)^{n-\kappa}(a_2 x_2)^{n-\kappa} \cdots (a_\kappa x_\kappa)^{n-\kappa}(a_{\kappa+1} x_{\kappa+1})^{n-\kappa} = 0.$$

For assigned values of $x_1, x_2, \cdots, x_\kappa$, $n-\kappa$ values of $x_{\kappa+1}$ are determined, the equation $I_{\kappa, n-\kappa} = 0$ being satisfied by any $\kappa+1$ values of the whole set of n. Such a set of n values with reference to a *general* symmetric form may be called an "involutive set." The object of Dr. Coble's paper is to determine the number of such involutive sets possessed by the general symmetric form and to determine the conditions upon the form in order that this number shall increase until it attains its maximum, ∞^κ.

A geometrical representation of the form as a spread of order $\kappa+1$ and dimension $n-\kappa-1$ in an $S_{n-\kappa}$ is given. The involutive sets then appear as "apolar n-points" of the spread which lie on a norm-curve $N_{n-\kappa}$. The definition of an apolar n-point is a set of n points any $\kappa+1$ of which are apolar to the spread of order $\kappa+1$.

F. N. COLE,
Secretary.

THE WINTER MEETING OF THE CHICAGO SECTION.

THE twenty-fourth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at the University of Chicago on Friday and Saturday, January 1–2, 1909, extending through three sessions. The total attendance was fifty, including the following forty-three members of the Society:

Mr. W. H. Bates, Professor G. A. Bliss, Dr. G. D. Birkhoff, Mr. H. E. Buchanan, Mr. Thomas Buck, Professor W. H. Butts, Professor H. E. Cobb, Dr. A. R. Crathorne, Professor D. R. Curtiss, Professor L. E. Dickson, Dr. E. L. Dodd, Mr. Arnold Dresden, Professor A. B. Frizell, Professor J. W. Glover, Professor A. G. Hall, Mr. T. H. Hildebrandt, Professor T. F. Holgate, Professor O. D. Kellogg, Dr. A. C. Lunn, Dr. E. B. Lytle, Mr. H. F. MacNeish, Dr. W. D. MacMillan, Professor J. L. Markley, Mr. E. J. Miles, Professor G. A. Miller, Professor E. H. Moore, Dr. R. L. Moore, Professor F. R. Moulton, Dr. L. I. Neikirk, Professor Alexander Pell, Mrs. Anna Pell, Professor C. A. Proctor, Mr. A. R. Schweitzer, Miss Ida M. Schottenfels, Professor G. H. Scott, Professor J. B. Shaw, Mr. C. G. Simpson, Professor H. E. Slaught, Dr. A. L. Underhill, Professor E. B. Van Vleck, Professor E. J. Wilczynski, Professor J. W. Young, Professor J. W. A. Young.

The chairman of the Section, Professor G. A. Miller, presided at the afternoon sessions on Friday and Saturday, and Professor E. B. Van Vleck, Vice-President of the Society, at the session on Saturday morning. The session on Saturday afternoon extended to a late hour in order to complete the reading of the papers. The Section has now found it necessary to devote two full days to its regular meetings, and the dinner on the evening of the first day has become an important feature, not only for the promotion of acquaintance, but also as affording an opportunity for the informal discussion of topics indirectly related to the interests of mathematicians.

On this occasion the subject was "The methods of appointment of university professors in foreign countries." The dis-

cussion was introduced by Professor Wilczynski, and participated in by Professors Bliss, Curtiss, Dickson, Hall, Holgate, Glover, Miller, Moore, and Van Vleck. This question was again taken up for formal consideration at the close of the session on Saturday morning, and the following resolution was presented by Professor Wilczynski and adopted: Resolved that the Chairman of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY appoint a committee of five to investigate the possibility of improving the character of the mathematical appointments in our colleges and universities, and to make recommendations to the Section at its next meeting. The Chairman appointed the following members of this committee: Professors E. J. Wilczynski, L. E. Dickson, T. F. Holgate, A. G. Hall, and E. B. Van Vleck.

The following officers of the Section were elected for the ensuing year: Professor G. A. Miller, Chairman, Professor H. E. Slaught, Secretary, and Professor O. D. Kellogg, third member of the programme committee.

It was announced that a limited number of the reprints from *Science* of the papers read last year at the joint meetings of mathematicians and engineers are available for distribution among the members of the Society, and will be supplied in the order of application while they last.

The following papers were read at this meeting:

1. Professor E. J. WILCZYNSKI: "Projective differential geometry, fifth memoir."

2. Mr. E. J. MILES: "Determination of the constants in Euler's problem concerning the minimum area between a curve and its evolute."

3. Mr. E. J. MILES: "Surfaces of revolution of minimum resistance."

4. Professor J. W. GLOVER: "A method of analysis of population and vital statistics."

5. Professor G. A. BLISS: "Some exceptional cases in implicit function theory."

6. Professor L. E. DICKSON: "General theory of modular invariants."

7. Professor A. B. FRIZELL: "Common complex numbers as an application of the theory of abstract groups."

8. Dr. A. C. LUNN: "The figures of equilibrium of rotating fluids."

9. Dr. A. C. LUNN: "Some notes on vector analysis."

10. Dr. A. C. LUNN: "The apparent size of a closed curve."

11. Mr. H. E. BUCHANAN: "On certain determinants connected with a problem in celestial mechanics."

12. Mr. H. E. BUCHANAN: "Periodic orbits of three finite bodies" (preliminary report).

13. Professor J. B. SHAW: "Qualitative algebra, second paper."

14. Mr. W. H. BATES: "The Kronecker invariant K of R_λ contained in R_n."

15. Dr. A. L. UNDERHILL: "Note on the calculus of variations."

16. Mr. A. R. SCHWEITZER: "The quaternion as an operator in Grassmann's extensive algebra, second paper."

17. Mr. W. W. DENTON: "On the osculating quartic of a plane curve."

18. Professor E. H. MOORE: "Note on a form of general analysis."

19. Professor J. W. YOUNG: "The notion of a general point field" (preliminary communication).

20. Professor L. E. DICKSON: "On the representation of numbers by modular forms."

21. Professor A. B. FRIZELL: "On the permutations of a series belonging to the second ordinal class."

Mr. Denton's paper was communicated to the Society and read by Professor Wilczynski. Professor Dickson's second paper was read by title. Mr. Buchanan's first paper appeared in full in the February BULLETIN. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the above list.

1. In this paper Professor Wilczynski takes up again the subject of the osculating conics of the plane sections of a surface, deducing some further consequences which were not mentioned in his first paper on this subject at the summer meeting of the Society. He also discusses the Steiner surface which has fourth order contact with a given surface at an arbitrary point. He wishes to emphasize a fact which he then ignored, viz., that one of the principal results of this paper, which is of fundamental importance for projective differential geometry, seems to have been discovered at least twice before and then again lost to the knowledge of mathematicians in general. The theorem

analogous to Meusnier's, which states that the osculating conics of those plane sections of a surface whose planes have a fixed line element in common with the surface form a quadric, was in fact discovered by Moutard in 1863. In 1880 Darboux independently obtained the same theorem, which had in the meantime been forgotten. But this theorem does not seem to be mentioned in any of the larger treatises on the theory of surfaces, nor in the Encyklopädie. The proof given by Professor Wilczynski closely resembles that of Darboux, but is of greater significance owing to the fact that his coordinates have a known geometrical meaning while those of Darboux are merely defined analytically.

2. Given two directed lines and a point A on one of them, the region is determined in which a second point B of the other line must lie, in order that the two points A and B may be joined by a cycloid tangent to the given lines at A and B. Mr. Miles in his first paper considers the following cases: (a) when no cusp of the cycloid is included between the points, (b) when one cusp is included, (c) when n cusps are included. The paper will be offered for publication in the *Annals of Mathematics*.

3. In the second paper of Mr. Miles the surfaces of revolution of minimum resistance, resulting from the resistance laws of Lõssl and Duchemin are obtained, and their similarity to the newtonian surface noted. A general law of resistance is then stated which gives surfaces of the same general characteristics as the preceding.

4. Professor Glover develops a method of analysis of population and vital statistics, to determine the effect of tuberculosis and other diseases on the mortality rate, and to estimate the present value of the corresponding monetary loss. Application is made with special reference to the city of Chicago.

5. The nature of the solutions $u = u(x, y)$, $v = v(x, y)$ of a pair of equations

$$x = \phi(u, v), \quad y = \psi(u, v),$$

in the neighborhood of a point where the functional determinant $\Delta = \partial(u, v)/\partial(x, y)$ does not vanish, is well known. If the curve $\Delta = 0$ in the u, v-plane has no singular points, the

singular points of the solutions can be classified in a few simple types. In the paper of Professor Bliss this classification is made, and the character of the functions $u(x, y)$ and $v(x, y)$ is discussed.

6. In this paper on modular invariants of one or more forms of any degrees, Professor Dickson avoids the complication which results from the application of linear transformations, or the equivalent annihilators, to functions of the coefficients in constructing invariants. Transformations are employed only to furnish a complete set of non-equivalent classes of the ground forms. Then a polynomial is an invariant if and only if it takes the same value for all the forms in a class. It is shown that the number of linearly independent absolute invariants equals the number of classes under the total group G; that the number of linearly independent invariants, including both absolute and relative, equals the number of classes under the group of transformations of determinant unity. Questions of linear independence are more fundamental in the theory of modular invariants than questions of complete independence. The general theory is applied to the determination of all the invariants of the general m-ary quadratic form in the $GF[p^n]$, there being $2m - 2 + p^n$ linearly independent invariants if $p > 2$; also to the invariants of the binary cubic, which has numerous modular invariants other than powers of its discriminant. For the practical construction of invariants, the general theory is supplemented by a uniform process, of function-theoretic nature, for the conversion of non-invariantive characterizations of the classes into formal invariants. It seems probable that modular invariants are destined to play a rôle in the theory of numbers commensurate with that played by algebraic and differential invariants in other branches of mathematics. The paper has been offered for publication in the *Transactions*.

7. Theoretical arithmetic considers two fundamental rules of combination, one of which is distributive with respect to the other. The natural numbers may be described as a set of symbols which form an abelian semigroup with respect to each rule, the first semigroup possessing a modulus, and which are well ordered by the second rule. The absolute numbers are a set of symbols which form a group with respect to the first rule, a semigroup with respect to the second, and a simply ordered

class with respect to both rules. The real numbers form a group with respect to the second rule; the real numbers excluding zero form a group with respect to the first rule; the set of all real numbers forms a simply ordered class with regard to the second rule.

In this paper Professor Frizell proves the following proposition: Given two rules of combination, of which one is distributive relatively to the other, and given a set of symbols which form a group and a simply ordered class with respect to the second rule, while if we omit the modulus of this group they form a group with respect to the first rule, then it is possible, by the introduction of a single new symbol i, to lay down two new rules, of which the first shall be distributive with respect to the second and which, when applied to the given symbols α, shall coincide respectively with the given rules, in such a way that the whole set of combinations $\alpha \cdot i$ and $(\alpha, \beta \cdot i)$ shall form a group with respect to the second rule and, if we omit its modulus, a group with respect to the first rule. The new set of compound symbols, however, is no longer simply ordered according to either rule.

8. Dr. Lunn's first paper gives a theorem related to those of Hamy and Volterra, showing that a mass of rotating fluid could not be in equilibrium, with its mass so distributed that the internal equipotential surfaces are confocal ellipsoids, even if the angular velocity varies with the distance from the axis.

9. Dr. Lunn's second paper gives: (*a*) a direct deduction of Rodrigues's parameter representation of an orthogonal matrix by integration of the vector differential equation defining the distribution of velocity in a rigid body; (*b*) a proof that a solenoidal vector can in general be represented as the vector product of two potential vectors; (*c*) some formulas on the vector potential analogous to those for the scalar potential, especially relating to functions of points and lines analogous to Green's function.

10. Dr. Lunn's third paper gives a formula for the solid angle subtended at a given point by a closed curve in space, as the scalar line integral around the curve of a suitably determined vector point function. The corrections for certain possible discontinuities are discussed and a number of examples worked out.

12. In his second paper Mr. Buchanan considers periodic orbits of three finite bodies near the lagrangean straight line orbits. Certain convenient parameters are introduced, the characteristic exponents are found and the linear terms are discussed. It is expected in a future report to give a full discussion of the problem.

13. Professor Shaw's paper discusses general multiplication. The subdivisions of the subject are (1) Lineal multiplication, which is independent of the unit system. There are three classes of laws: (*a*) those depending on limitation types;* (*b*) those due to invariant expressions in symmetric groups, expressions corresponding to the single units in the quadrates of order 1 in the Frobenius algebra corresponding to the group;† (*c*) each limitation type determines a subgroup of $G_{n!}$, and each subgroup has symmetric expressions, corresponding to the single units in the quadrates of order 1 in its Frobenius algebra. Examples are Grassmann's and Gibbs's systems. (2) Orthogonal multiplication, which is not altered by orthogonal changes of the units. Examples are McAulay's multernions.‡ (3) Linear homogeneous multiplication, which is not altered by linear homogeneous transformations of the units. Examples are given. (4) Generalized multiplication, which is independent of the units. The paper will be offered for publication in the *Transactions*.

14. Professor Maschke, in the *Transactions*, volume 6, pages 87–93, calculated, for even values of λ, the Kronecker invariant K of a space of λ dimensions R_λ, represented as a differential parameter of a space of higher dimensions R_n containing R_λ. At the close of this calculation, he says that the principles which lead to this expression of K for even values of λ will doubtless also be sufficient to solve the more complicated problem for the case of odd values of λ. In the present paper, Mr. Bates calculates the expression of K for odd values of λ.

15. The solution of the Jacobi equation, in connection with the minimizing of

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* Shaw, Synopsis of Linear Associative Algebra, p. 77.

† Ibid., pp. 49–51.

‡ *Proc. Royal Soc. of Edinburgh*, 1907–8, vol. 28, pp. 503–585. Joly: *Proc. Royal Irish Acad.*, 1897 (3) 5, pp. 73–123.

$$J = \int_{t_0}^{t_1} F(x, y, x', y')dt,$$

may present difficulties. In volume 9, page 336, of the *Transactions*, and in a note presented to the Chicago Section, December, 1907, methods were given wherewith one may avoid the solution of the Jacobi equation, and yet obtain information as to the extent of the interval in which a weak minimum exists. In the present note Dr. Underhill applies these methods to several of the standard problems.

16. Mr. Schweitzer defines the vectors

$$\begin{aligned} X &= x_1E_1 + x_2E_2 + x_3E_3 + x_4E_4, \\ i_3X &= \lambda_xE_1 + \mu_xE_2 + \nu_xE_3 + \omega_xE_4, \\ j_3X &= \lambda'_xE_1 + \mu'_xE_2 + \nu'_xE_3 + \omega'_xE_4, \\ k_3X &= \lambda''_xE_1 + \mu''_xE_2 + \nu''_xE_3 + \omega''_xE_4, \end{aligned}$$

where

$$\begin{aligned} \xi_x &= \xi_1x_1 + \xi_2x_2 + \xi_3x_3 + \xi_4x_4, \\ \xi &= \lambda, \lambda', \lambda''; \ \mu, \mu', \mu''; \ \nu, \nu', \nu''; \ \omega, \omega' \omega''; \end{aligned}$$

and expresses

$$\begin{aligned} i_3X &= i_2X + j''_1X + k''_1X, \\ j_3X &= j_2 + k'_1X + i'''_1X, \\ k_3X &= k_2X + i''_1X + j'''_1X \end{aligned}$$

by suitable definitions, and then determines the operators i_3, j_3, k_3 in the form

$$\begin{aligned} i_3 &= \lambda_2i_{\epsilon_2} + \lambda_1 + \lambda_3j_{\epsilon_{11}} + \lambda_4k_{\epsilon_{12}}, \\ j_3 &= \lambda'_2j_{\epsilon_2} + \lambda'_1 + \lambda'_3k_{\epsilon_{11}} + \lambda'_4i_{\epsilon_{12}}, \\ k_3 &= \lambda''_4k_{\epsilon_2} + \lambda''_1 + \lambda''_2i_{\epsilon_{11}} + \lambda''_3j_{\epsilon_{12}}, \end{aligned}$$

where

$$\epsilon_2^2 = \epsilon_{11}^2 = \epsilon_{12}^2 = 1,$$

by the specification that the outer products $[X \cdot i_2X \cdot j_2X \cdot k_2X]$, $[X \cdot j''_1X \cdot k'_1X \cdot i''_1X]$, $[X \cdot k''_1X \cdot i'''_1X \cdot j'''_1X]$ shall be identically equal respectively to

$$\begin{aligned} &\rho_2(x_1^2 + x_2^2 + x_3^2 + x_4^2)^2E_1E_2E_3E_4, \\ &\rho'_1(x_1^2 + x_2^2 + x_3^2 + x_4^2)^2E_1E_2E_3E_4, \\ &\rho''_1(x_1^2 + x_2^2 + x_3^2 + x_4^2)^2E_1E_2E_3E_4, \end{aligned}$$

ρ_2, ρ_1', ρ_1'' being constants. The operators $i_{\epsilon_2}, j_{\epsilon_2}, k_{\epsilon_2}$; $i_{\epsilon_{11}}, j_{\epsilon_{11}}, k_{\epsilon_{11}}$; $i_{\epsilon_{12}}, j_{\epsilon_{12}}, k_{\epsilon_{12}}$ constitute (with unity) three systems of hamiltonian units. The systems have the "same sense" if $\epsilon_2 = \epsilon_{11} = \epsilon_{12}$ and all the systems are "right-handed" if $\epsilon_2 = \epsilon_{11} = \epsilon_{12} = -1$. Thus the function $\rho(x_1^2 + x_2^2 + x_3^2 + x_4^2)^2$ may be said to be *characteristic* of the quaternion as an operator in Grassmann's extensive algebra.

(17) The projective differential geometry of a plane curve which is not a straight line is equivalent to the theory of the invariants and covariants of the linear homogeneous differential equation of the third order

$$\frac{d^3y}{dx^3} + 3p_1\frac{d^2y}{dx^2} + 3p_2\frac{dy}{dx} + p_3y = 0.$$

Mr. Denton has computed the equation of the osculating quartic, *i. e.*, the quartic having contact of the highest possible order with the given curve in a given non-singular point. An invariant triangle which was first introduced by Professor Wilczynski is used as triangle of reference. The calculation, which was long and laborious, was carried out completely by two independent methods, thus furnishing an adequate check for the final explicit equation. The paper will be offered for publication in the *Transactions*.

(18) The note of Professor Moore, to be published in the BULLETIN, gives a brief synopsis of his lectures, "On the theory of bilinear functional operations," delivered at the New Haven Colloquium, September, 1906, together with an account of subsequent developments and applications of the central thought of those lectures.

(19) Professor Young's paper describes what he calls a general point field, which is the geometric counterpart of the notion of a general number field. A net of rationality * and a chain in n-dimensions are special types of point fields of n-dimensions. A point field on a line is a class of points isomorphic with a number field; i. e., if three points of the field are labelled 0, 1, ∞ respectively, and the corresponding number system developed,† the numbers corresponding to the points of the given field form a field (of numbers). The synthetic definition of such fields is discussed. Fields of n-dimen-

sions are defined in terms of linear fields, and applications to various analytic problems are indicated. A general theory of linear fields defined by three points is discussed under general hypotheses.

(20) The second paper by Professor Dickson proves for the cases $m \leqq 3$ that every form of degree m in $m+1$ variables vanishes for values, not all zero, in any finite field. The larger part of the paper relates to forms in two or more variables which represent exclusively cubes in a finite field. The paper will appear in the BULLETIN.

(21) If a series of type ω be re-arranged by bringing each element in succession to the first place and then treating the remainder of each new series in the same way, the resulting permutations form a series of type $\omega_1 = \omega^\omega$. Now the set of all conceivable arrangements of the series of natural numbers has the cardinal of the continuum, but every possible arrangement of the natural numbers forms a series belonging to the second class of ordinal numbers. Professor Frizell shows, in his second paper, how to exhibit the set of natural numbers as an ordinal type higher than ω_1, and proves that it is not possible to get outside the second ordinal class by permuting the elements of any series of type lower than Ω.

H. E. SLAUGHT,
Secretary of the Section.

* Cf. Veblen and Young, "A set of assumptions for projective geometry." *Amer. Jour. of Mathematics*, vol. 30 (1908), p. 356ff.

† Veblen and Young, loc. cit., p. 352.

THE SIXTIETH MEETING OF THE AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE.

THE sixtieth meeting of the American Association for the Advancement of Science was held at Johns Hopkins University during the convocation week, December 28, 1908, to January 2, 1909. The president of the meeting was Professor T. C. Chamberlin, University of Chicago. The address of the retiring president, Professor E. L. Nichols, entitled "Science and the practical problems of the future," was given at the Peabody Institute on the evening of the opening day.

Comparatively few papers on pure mathematics appeared on the program of Section A because of the fact that the AMERICAN MATHEMATICAL SOCIETY held its annual meeting in affiliation with the Association. The address of the retiring vice-president, President E. O. Lovett, Rice Institute, Houston, Texas, was read by the secretary of the section. It was entitled "The problem of several bodies, recent progress in its solution," and an abstract of it has appeared in a recent number of *Science* (January 15, 1909).

The officers of the section were: vice-president, C. J. Keyser; secretary, G. A. Miller; councilor, G. B. Halsted; member of the general committee, F. R. Moulton; sectional committee, E. O. Lovett, C. J. Keyser, G. A. Miller, E. B. Frost, Harris Hancock, F. R. Moulton, E. W. Brown. On the recommendation of the sectional committee the following thirty-six members of the AMERICAN MATHEMATICAL SOCIETY were elected fellows of the Association: Joseph Allen, R. B. Allen, Oskar Bolza, W. H. Bussey, B. E. Carter, Abraham Cohen, E. H. Comstock, H. A. Converse, S. A. Corey, F. F. Decker, C. C. Engberg, F. C. Ferry, William Gillespie, C. O. Gunther, U. S. Hanna, A. E. Haynes, W. J. Hussey, Kurt Laves, A. H. McDougall, Max Mason, W. F. Osgood, J. M. Page, M. T. Peed, James Pierpont, W. J. Rush, P. L. Saurel, G. T. Sellew, E. B. Skinner, D. E. Smith, P. F. Smith, R. P. Stephens, H. D. Thompson, E. B. Van Vleck, Oswald Veblen, H. S. White, F. S. Woods.

In addition to the address of the vice-president, the following sixteen papers were read before the section:

1. Mr. V. M. SLIPHER: "The spectrum of Mars."

2. Professor E. B. FROST and Mr. J. A. PARKHURST: "Spectrum of comet Morehouse."

3. Professor E. E. BARNARD: "On the changes in the tail of comet Morehouse."

4. Professor FRANK SCHLESINGER: "The orbit of the Algol type variable Delta Libræ."

5. Professor MILTON UPDEGRAFF: "The 6-inch transit circle of the U. S. Naval Observatory."

6. Professor F. R. MOULTON: "On certain implications of possible changes in the form and dimensions of the sun, and some suggestions for explaining certain phenomena of variable stars."

7. Mr. R. H. BAKER: "On the spectra of Alpha Virginis and similar stars."

8. Mr. F. C. JORDAN: "The orbit of Alpha Coronæ Borealis."

9. Professor E. B. FROST: "Radial velocities in Professor Boss's star stream in Taurus."

10. Messrs. PHILIP FOX and GEORGIO ABETTI: "The interaction of sun spots."

11. Professor HARRIS HANCOCK: "Elliptic realms of rationality."

12. Dr. ARTEMAS MARTIN: "Algebraic solution of the 'three point' problem."

13. Professor J. B. WEBB: "Esperanto and a sexdecimal notation."

14. Professor J. A. MILLER and Mr. W. R. MARRIOTT: "Comet Morehouse."

15. Dr. L. A. BAUER: "On the interpolation formula of geophysics."

16. Mr. H. W. FISK: "A graphical aid to the determination of latitude and azimuth from Polaris observation."

In the absence of their authors the paper by Mr. Slipher was read by J. A. Miller; the joint paper by E. B. Frost and J. A. Parkhurst was presented by W. S. Eichelberger; the three papers by Frank Schlesinger, E. H. Baker and F. C. Jordan, respectively, were read by J. A. Brashear; F. R. Moulton's paper was read by E. D. Roe; G. F. Hull presented the two papers by E. B. Frost and by Philip Fox and Georgio Abetti, the papers by E. E. Barnard and Artemas Martin were read by title. The remaining papers were read by their authors. The

following abstracts of those which were of mathematical interest bear numbers corresponding to those of the titles in the list. The abstracts of the others appeared in the report of Section A of this meeting published in *Science*, January 22, 1909.

6. The problems treated in the paper by Professor Moulton are: (1) The theoretical shape of the sun, (2) the character and period of its possible gravitational oscillations, (3) the effects of changes of its dimensions upon its rate of rotation, (4) its energy of rotation, (5) its potential energy, (6) its temperature and rate of rotation, and (7) applications of the same ideas to variable stars.

The results are: (1) The sun is oblate and the theoretical difference in its polar and equatorial diameters is less than 0″.01. (2) Its gravitational oscillations are expressible in spherical harmonics whose periods depend upon their order. Assuming the sun to be a homogeneous liquid, the longest period is 3 h. 8 m. If it has the viscosity of water, this oscillation will change to 37 per cent. of its value in 2.2×10^{15} years. (3) The change of the sun's diameter by 0″.1 will change its period of rotation by 7.8 minutes. (4) The formula was found for the change in the rotational energy. (5) The formula for the potential of a spheroid of polar radius c, equatorial $c\sqrt{1+\lambda^2}$, and mass m upon itself is

$$V = \tfrac{3}{5}k^2\frac{m^2}{c}(1 - \tfrac{1}{3}\lambda^2 - \tfrac{7}{15}\lambda^4 - \cdots).$$

(6) The expansion of the sun by 0″.1 will decrease its temperature (assuming its specific heat is unity) more than 1400° C., and if it obeys Stefan's law, diminish its radiation (assuming its temperature to be 6000° before expansion) by more than 65 per cent. (7) It is shown how gravitational oscillations can explain many puzzling phenomena of variable stars, such as variable periods in the so-called eclipse variables, secondary maxima and minima, varying maxima and minima, etc. It is thought that these factors are supplementary even in those cases where the binary character of the star is certain, and that perhaps in certain classes of stars they may be the only causes of variability.

11. It is known that on the Riemann surface associated with

$$s = \pm\sqrt{A(z-a_1)(z-a_2)(z-a_3)(z-a_4)},$$

in which a_1 and a_2 are connected by a canal as are also a_3 and a_4, every one-valued function of position which has everywhere a definite value is of the form $w = p + q \cdot s$, where p and q are rational functions of the complex variable z; and reciprocally every function of this form is a one-valued function of position on this Riemann surface. If we denote two such functions by

$$w_1 = p_1 + q_1 \cdot s, \quad w_2 = p_2 + q_2 \cdot s,$$

then the sum, difference, product and quotient of the two functions w_1 and w_2 are functions of the form $w = p + q \cdot s$.

Let z take all complex values and consider the collectivity of all rational functions of z with arbitrary constant real or complex coefficients. These functions form a closed realm, the individual functions of which repeat themselves through the processes of addition, subtraction, multiplication and division, since clearly the sum, the difference, the product and the quotient of two or more rational functions is a rational function and consequently an individual of the realm. This realm is denoted by (z).

It is evident that if we add (or *adjoin*) the algebraic quantity s to this realm we will have another realm, the individual functions or elements of which repeat themselves through the processes of addition, subtraction, multiplication and division. This realm includes the former realm. We shall call it the elliptic realm and denote it by (s, z).

By a theorem due to Liouville, the most general one-valued doubly periodic function is a rational function of z and s. It is consequently a one-valued function of position on the Riemann surface and belongs to the elliptic realm of rationality (z, s).

The elliptic or doubly periodic realm of rationality (z, s) degenerates into the simply periodic realm when any pair of branch points are equal, say $a_1 = a_2$; and into the realm of rational functions when two pairs of branch points are equal, say $a_1 = a_2$ and $a_3 = a_4$.

Thus the elliptic realm includes the three classes of one-valued functions: (1) the rational functions, (2) the simply periodic functions, (3) the doubly periodic functions. All these one-valued functions, and only these, have algebraic addition theorems.

In other words, *all functions of the realm* (z, s) *have algebraic addition theorems, and no one-valued function that does not belong to this realm has an algebraic addition theorem.*

We have thus the theorem:
The one-valued functions of position on the Riemann surface

$$s^2 = A(z - a_1)(z - a_2)(z - a_3)(z - a_4)$$

belong to the closed realm (z, s) *and all elements of this realm and no others have algebraic addition theorems.*

Professor Hancock's paper will be offered to the *American Journal of Mathematics* for publication.

12. The paper by Artemas Martin is devoted to an algebraic determination of the point within a triangle at which the sides subtend given angles. The paper is to appear in the *Mathematical Magazine*, edited by the author.

13. The paper by J. Burkitt Webb is devoted to exhibiting the advantages which would result from the adoption of a system of notation with 16 as its base. The success which has attended the movements towards a universal language has inspired the author with hope in the success of a movement towards the selection of a more useful system of notation, and he pointed out the many advantages which the base 16 would offer.

15. The rather prevalent custom of resolving or expressing every natural phenomenon — be it periodic or otherwise — by a Bessel or a Fourier series, or by spherical harmonic functions, has brought about at times, especially in geophysical and cosmical phenomena, if not direct misapplications, at least misinterpretations of the meaning and value of the derived coefficients. Instead of clarifying the situation our calculations may have actually contributed to befog it. Instead of rejecting, one must learn to consider the outstanding residuals as the *true* facts of nature and not treat them as though they were "abnormal" or contrary to nature's law.

Dr. Bauer exemplified these statements in a brief discussion of two cases that are typical in geophysical investigations — the one involving an application of spherical harmonic functions to the representation of the distribution of the earth's magnetism over the earth, while the other involved the use of Fourier series in the representation of certain diurnal geophysical phenomena.

The chief purpose of the paper was to call renewed attention to the limitations, from a physical standpoint, of the form

of "interpolation formulas" usually employed in the representation of natural phenomena. The formula of Gauss is simply a mathematical approximation to the actual law of distribution. In spite of all the work done, no one up to date has succeeded in giving any physical interpretation of the coefficients of the harmonic terms beyond those of the first order involving three coefficients, and these three stand for the simplest possible case of a first approximation to the actual state of the earth's magnetism: that of a uniform magnetization about a diameter inclined to the axis of rotation.

16. Mr. H. W. Fisk considers the formula for latitude,

$$\phi = h - p \cos t + \tfrac{1}{2} p^2 \sin 1' \sin^2 t \tan h + \cdots$$

from Chauvenet's Astronomy I., § 176, and the formula for azimuth,

$$A = p \sin t \sec \phi + \tfrac{1}{2} p^2 \sin 1' \sec \phi \tan \phi \sin 2t + \cdots$$

from Jordan, "Zeit und Orts-Bestimmung," page 122. The first terms of these formulas are readily computed. The last terms, called correction terms, are arranged as a set of curves from which the value is quickly taken by inspection. The geographical limits within which this method may be used, as well as the expected accuracy under different conditions are discussed. Attention is given to the change in correction terms due to the progressive change in the value of p. Assuming, for instance, that p will continue indefinitely to change at its present rate 0′.34 per annum, we may find the period during which the curves could be used within the error 0′.1. The latitude curves are found to be good 83 years at latitude 10° and about 8 years at latitude 60°. The azimuth curves are correspondingly useful for 80 years at 10° and 4 years at 60°.

The next regular meeting of the association will be held in Boston under the presidency of President D. S. Jordan, Stanford University. No summer meeting will be held during 1909, in order to permit the members to attend the meetings of the British Association at Winnipeg, which has extended a cordial official invitation to the members of the American Association. Professor E. W. Brown, Yale University, was elected vice-president and chairman of Section A, and Professor G. A. Miller, University of Illinois, continues as secretary. The

section elected Professor Winslow Upton, Ladd Observatory, as member of the sectional committee for five years.

G. A. MILLER,
Secretary.

SOME SURFACES HAVING A FAMILY OF HELICES AS ONE SET OF LINES OF CURVATURE.*

BY MISS EVA M. SMITH.

IN a recent paper, Forsyth † gives a general method for the determination of surfaces with assigned lines of curvature, and he solves completely the case where both sets are circles. We apply the method to the case where one of the given sets consists of helices, and it appears that surfaces do exist having as one set of lines of curvature general helices ($\rho/\tau =$ constant along each curve), but there are always limitations on the forms of ρ and τ. In particular, ρ and τ cannot both be constant along every curve. The complete solution seems to be too wide for analytic discussion, but there are two particular cases for which definite results can be obtained. This note contains a discussion of these cases.

1) Assuming that ρ and τ are constant along each curve of the set (regular helices), we obtain the result that: *There are no surfaces with regular helices as one set of lines of curvature.*

2) If ρ/τ is constant along each curve of one set of lines of curvature, and the other set consists of geodesics, we can obtain a complete solution; the equations of the resulting surfaces in parametric form are given at the end of this paper. The notation and equations used are those given in Darboux, Théorie des surfaces, volume 2, but derivatives with respect to u and v are here denoted by suffixes 1 and 2 respectively.‡

§ 1.

Consider the case where the helices are all regular. The curves $v =$ constant are helices, and therefore ρ and τ for these curves are functions of v only, and we denote τ/ρ by k.

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* For the suggestion of this subject I am indebted to Prof. A. R. Forsyth.

† *Messenger of Mathematics*, vol. 38 (1908), pp. 33–44.

‡ Note that p_1 is not $\partial p/\partial u$. To express derivatives of the rotations we use parentheses, *e. g.* $(p)_1 = \partial p/\partial u$.

We have the following set of equations* :

$$p = 0, \quad q_1 = 0,$$

$$(1) \quad \frac{1}{\tau} = \frac{1}{A}\phi_1,$$

$$(2) \quad r = \frac{\sin\phi}{\rho}A, \qquad (2') \quad r = -\frac{1}{C}A_2,$$

$$(3) \quad q = -\frac{\cos\phi}{\rho}A, \qquad (4) \quad p_1 = \frac{1}{r}(q)_2,$$

$$(5) \quad r_1 = -\frac{1}{q}(p_1)_1, \qquad (5') \quad r_1 = \frac{1}{A}C_1,$$

where ϕ is the angle between the osculating plane of the curve $v =$ constant and the normal plane to the surface. From (2), (3), (4), (5) we obtain

$$(7) \quad r_1 = 2\frac{\phi_{12}}{k\phi_1}\sec\phi + \frac{k_2}{k^2}\operatorname{cosec}^2\phi\sec\phi - \frac{1}{k\phi_1}\left(\frac{\phi_{12}\operatorname{cosec}\phi}{\phi_1}\right)_1$$

Also from (1), (2'), (5'),

$$(8) \quad r_1 = \frac{1}{k}\left[\frac{\tau_2}{\tau}\cot\phi\operatorname{cosec}\phi - \frac{1}{\phi_1}\left(\frac{\phi_{12}\operatorname{cosec}\phi}{\phi_1}\right)_1\right].$$

From (7) and (8)

$$\frac{\phi_{12}}{\phi_1} = \tfrac{1}{2}\left(\frac{\tau_2}{\tau}\cot^2\phi - \frac{k_2}{k}\operatorname{cosec}^2\phi\right).$$

Differentiating with respect to u and substituting in (8), we get

$$(9) \quad r_1 = \tfrac{3}{2}\cot\phi\operatorname{cosec}\phi\left[\operatorname{cosec}^2\phi\left(\frac{\tau_2}{\tau} - \frac{k_2}{k}\right) + \frac{\tau_2}{\tau}\right].$$

Combining (9) with (6) we obtain, after some reduction,

$$6\operatorname{cosec}^4\phi\left(\frac{\tau_2}{\tau} - \frac{k_2}{k}\right) - \tfrac{3}{2}\operatorname{cosec}^2\phi\left(\frac{\tau_2}{\tau} - \frac{3k_2}{k}\right) + \frac{k^2}{2}\left[\operatorname{cosec}^2\phi\left(\frac{\tau_2}{\tau} - \frac{k_2}{k}\right) - \frac{\tau_2}{\tau}\right] = 0.$$

* Darboux, l. c., pages 383, 386.

This equation determines ϕ as a function of v only, but as the curves $v =$ constant are not plane curves, ϕ is a function of both u and v, hence the equation must reduce to an identity. Equating to zero the coefficients of the various powers of cosec ϕ, we get

$$\frac{\tau_2}{\tau} = 0, \quad \frac{\tau_2}{\tau} - \frac{k_2}{k} = 0,$$

or $\rho =$ constant, $\tau =$ constant. Substituting in the preceding equations, we obtain finally $C = 0$. Hence no surfaces exist of the kind we seek.

§ 2

Let us now assume that $\rho/\tau = \alpha$, a function of v only, and that the curves $u =$ constant are geodesics. Having regard to the properties of geodesics, we may put $C = 1$.

The equations referred to above now become

$$p = 0, \quad q_1 = 0, \quad r_1 = 0,$$

(i) $r = \dfrac{\sin\phi}{\rho} A,$ (i′) $r = -A_2,$

(ii) $q = -\dfrac{\cos\phi}{\rho} A,$ (iii) $(r)_2 + qp_1 = 0,$

(iv) $p_1 = \dfrac{1}{r}(q)_2,$ (iv′) $p_1 = V.$*

There are two cases to be considered according as V is zero or not. If V is not zero, take a new variable t so that $Vdv = dt$. From (iii), (iv)

$$\frac{\partial^2 r}{\partial t^2} + r = 0, \quad r^2 + q^2 = U'^2,$$

$$\therefore r = U' \sin(t + U), \quad q = -U' \cos(t + U).$$

From (i), (ii)

$$t + U = \phi, \quad -A_2 = U' \sin(t + U).$$

We can absorb U' into A, which is equivalent to reducing U' to unity, and we obtain finally

* Capital letters U, V denote functions of the variables u alone, v alone respectively.

$$A = \rho, \quad C = 1, \quad \rho_2 = -\sin(t + U), \quad U = \alpha u.$$

The last result shows that α, which is a function of v only, must be an absolute constant; also ρ must be of the form $a \cos \alpha u + b \sin \alpha u$, where a and b are functions of v alone and

$$a_2 = -\sin t, \quad b_2 = -\cos t.$$

All the conditions are now satisfied, so that surfaces do exist having the required properties, and to obtain them we have the equations

$$x_1^2 + y_1^2 + z_1^2 = \rho^2, \quad x_1 x_2 + y_1 y_2 + z_1 z_2 = 0, \quad x_2^2 + y_2^2 + z_2^2 = 1.$$

$$D = \rho^2 \cos(t + ku), \quad D' = 0, \quad D'' = \rho V^*$$

These can be integrated and we obtain as coordinates of a point on a surface referred to a special set of axes

$$x = \rho_1 \frac{\sin(\sqrt{1+\alpha^2}.u)}{\sqrt{1+\alpha^2}} - \rho \cos \sqrt{(1+\alpha^2}.u),$$

$$y = \rho_1 \frac{\cos(\sqrt{1+\alpha_1^2}.u)}{\sqrt{1+\alpha^2}} + \rho \sin(\sqrt{1+\alpha^2}.u),$$

$$z = \frac{\rho_1}{\alpha\sqrt{1+\alpha^2}}.$$

From a further consideration of the equation given on pages 383, 386, l. c., it appears that the curves $u =$ constant are plane curves, whose radii of curvature are given by

$$1/\rho = p_1 = V.$$

This shows that the set $u =$ constant consists of the different positions of an invariable curve in a plane which moves in a direction perpendicular to itself.

If $V = 0$, we see at once that the surface is a developable whose coordinates may be written in the form

$$x = a + lv, \quad y = b + mv, \quad z = c + nv,$$

where l, m, n, a, b, c are all functions of u only, and in virtue

* Darboux, l. c., page 378.

of the special properties of the curves $u = \text{constant}, v = \text{constant}$, satisfy the following relations:

$$l^2 + m^2 + n^2 = 1, \quad l_1^2 + m_1^2 + n_1^2 = 1,$$

$$\frac{a_1}{l_1} = \frac{b_1}{m_1} = \frac{c_1}{n_1} = \beta, \text{ say.}$$

Then we have $A = v + \beta$ and $ds^2 = (v + \beta)^2\, du^2 + dv^2$. The equations to determine p, q, r, p_1, q_1, r_1, now become

$$p = 0, \quad q_1 = 0, \quad r_1 = 0, \quad p_1 = 0,$$

$$(1) \quad r = -1, \qquad r = \frac{\sin \phi}{\rho}(v + \beta),$$

$$(2) \quad q = U, \qquad (2') \quad q = -\frac{\cos \phi}{\rho}(v + \beta),$$

$$(3) \quad \tau\phi_1 = v + \beta.$$

Combining (1), (1'), (2), (2') we obtain

$$\tan \phi = \frac{1}{U},$$

which, upon differentiation with respect to u, and substitution from (3), leads to the following equation:

$$\alpha = \frac{\rho}{\tau} = \frac{U_1}{(1 + U^2)^{\frac{3}{2}}}.$$

From this it is apparent that ρ/τ, which is a function of v only, is an absolute constant. By integration we obtain

$$\frac{U}{\sqrt{1 + U^2}} = \alpha u + \gamma \qquad (\gamma = \text{constant}).$$

No generality is lost by making $\gamma = 0$, and then

$$U = \frac{\alpha u}{\sqrt{1 - \alpha^2 u^2}}.$$

The equations to be integrated are

$$\sum l^2 = 1, \quad \sum l_1^2 = 1, \quad \frac{l_1}{a_1} = \frac{m_1}{b_1} = \frac{n_1}{c_1} = \frac{1}{\beta},$$

$$\begin{vmatrix} l & m & n \\ l_1 & m_1 & n_1 \\ l_{11} & m_{11} & n_{11} \end{vmatrix} = U$$

and the surface is defined by the equations

$$x = a + kuv,$$

$$y = b + \frac{kuv}{U\sqrt{1-k^2}}(\cos\psi + k\sin\psi\ U),$$

$$z = c + \frac{kuv}{U\sqrt{1-k^2}}(\sin\psi - kU\cos\psi),$$

where a, b, c are perfectly determinate functions of u, and

$$k = \frac{\alpha}{\sqrt{1+\alpha^2}}, \quad \sin k\psi = \frac{ku}{\sqrt{1-k^2}}.$$

BRYN MAWR COLLEGE, PENNA.,
December, 1908.

NOTE ON ENRIQUES'S REVIEW OF THE FOUNDATIONS OF GEOMETRY.

BY MR. A. R. SCHWEITZER.

(Read before the Chicago Section of the American Mathematical Society, April 18, 1908.)

1. As is well known, a continuous descriptive space may be extended to a projective one by the suitable definition of the projective points. For the definition of euclidean geometry on the basis of descriptive-projective geometry, we distinguish between the proper and improper points of the space and the assumption is made that all the improper points lie on the unique improper plane. This assumption may be provided for by means of a

suitable descriptive axiom. The following question then arises:* Is it possible to define a unique euclidean geometry on the basis of the preceding descriptive-projective space without introducing further indefinables or axioms? In the present note I show that such a definition is possible if we employ the notion of a propositional function.†

2. The conventional manner of defining euclidean geometry on the basis of descriptive-projective geometry is to select an elliptic polar system, say Σ_0, in the unique improper plane and to define perpendicularity, congruence, etc., with reference to it. Every such elliptic polar system Σ_0, however, gives rise to a euclidean geometry $G(\Sigma_0)$. A typical congruence proposition in this geometry will have the form:‡ "If . . . with reference to Σ_0, then . . . with reference to Σ_0." Thus the class of elliptic polar systems $\{\Sigma_0\}$ in the improper plane gives rise to a unique class of euclidean geometries $\{G(\Sigma_0)\}$. The members of this latter class are, however, given by means of the geometry function $G(\Sigma)$ of a variable elliptic polar system Σ in the improper plane. This geometry function is then uniquely defined. A typical congruence proposition of $G(\Sigma)$ has the form: "If . . . with reference to Σ then . . . with reference to Σ." The terms of the latter implication are propositional functions of the variable Σ. As Russell maintains,§ such terms are not propositions but yield propositions for every constant value of the variable; the implication which involves the propositional functions as terms is, however, a proposition and with reference to the implication, in the terminology of Peano, Σ is an apparent variable.

* Compare Enriques, Encyklopädie der mathematischen Wissenschaften, vol. III, 1, p. 34.

† See B. Russell, The Principles of Mathematics, pp. 13, 82.

‡ The spaces indicated are to be filled out suitably by congruence properties.

§ l. c., p. 13. In this connection, see some interesting points brought out by a controversy between McColl and Russell in the journal *Mind*, n. s. vol. 17 (1908), pp. 151–152; 300–302.

NOTES.

THE opening (January) number of volume 31 of the *American Journal of Mathematics* contains the following papers: "The quadric spreads connected with the configuration $\Gamma^{n+2}_{n+4,n}$ and a special case in the Pascal hexagram," by W. B. CARVER; "The group membership of singular matrices," by A. RANUM; "Methods to determine the primitive roots of a number," by G. A. MILLER; "Standard forms of certain types of Peirce algebras," by J. B. SHAW; "On the law of gravitation in the binary systems," by F. L. GRIFFIN; "Lösung des Lehmer'schen Problems," by E. LANDAU.

THE next meeting of the Deutsche Mathematiker-Vereinigung will be held at Salzburg.

AT the meeting of the London mathematical society held on January 14, 1909, the following papers were read: By H. HILTON, "Canonical form of a linear substitution"; by J. HAMMOND, "On the solutions of the quintic"; by A. CUNNINGHAM, "On octavic and sexadecimic residuacity"; by E. W. HOBSON, "On change of the variable in a Lebesgue integral"; by F. H. JACKSON, "On Abel's extension of Taylor's series"; by H. M. MACDONALD, "Note on the evaluation of a certain integral containing Bessel's functions."

At the meeting of the society held on February 11 the following papers were read: By A. C. DIXON, "On the relation between Pfaff's problem and the calculus of variations"; by W. H. YOUNG, "On implicit functions and their differentials"; by W. H. SALMON, "On a certain family of cubic surfaces"; by E. W. HOBSON, "Some fundamental properties of Lebesgue integrals of a two dimensional domain"; by L. E. DICKSON, "Modular invariants of a general system of linear forms."

THE executive committee of the international commission of secondary instruction in mathematics held its first session at Cologne in September, 1908, and perfected its organization. The principal features are the following:

Each country having at least two delegates at two or more congresses of mathematicians will have one vote in the com-

mission; other countries may send one delegate, who will have no vote. The executive committee is final authority on all questions concerning the commission. The official organ will be *L'Enseignement Mathématique*. The purpose of the commission is to institute an inquiry and to prepare a comprehensive report concerning the aims and methods of mathematical instruction throughout the world. The commission hopes to present its complete report to the fifth international congress, to be held at Cambridge, England, in August, 1912.

THE Spanish association for the advancement of science was organized last year; the first meeting was held at Zaragossa, October 22–29, 1908. Sixteen papers were read in the section of mathematics. Two committees were appointed, one to examine and report on the feasibility of organizing a Spanish mathematical society, the other to study the formation of a Spanish mathematical vocabulary.

THE royal academy of sciences of Turin announces the following programme for the seventeenth award of the Bressa prize. The purpose of the competition is to reward the scientist or inventor, irrespective of affiliation or nationality, who, during the period 1907–1910, has in the judgment of the academy made the most brilliant and useful discovery or who has made the most noteworthy advances in physical or experimental sciences, natural history, pure or applied mathematics, chemistry, physiology or pathology, not excluding geology, history, geography, or statistics.

Competing memoirs should be printed and sent to the president with the statement that they are entered for the competition, which closes December 31, 1910. The net value of the prize is 9300 francs.

THE following courses in mathematics will be offered during the summer semester, 1909.

UNIVERSITY OF GÖTTINGEN. — By Professor F. KLEIN: Mechanics of the continuum, four hours; Seminar, two hours. — By Professor D. HILBERT: Theory of numbers, four hours; Selected problems in the theory of functions, two hours; Seminar, two hours. — By Professor C. RUNGE: Differential equations, four hours; Seminar, two hours. — By Professor W. VOIGT: Thermodynamics, four hours. — By Professor L.

PRANDTL: Theory of aeronautics, four hours. — By Professor A. SCHWARZSCHILD: Advanced celestial mechanics, four hours. — By Professor E. WIECHERT: Theory of potential, four hours. — By Professor E. ZERMELO: Differential and integral calculus, four hours. — By Dr. O. TOEPLITZ: Analytic geometry, four hours. — By Dr. P. KOEBE: Elementary mathematics from a higher standpoint, four hours.

DURING the year 1908 the following doctorates were conferred at the University of Paris with mathematics as the major subject (the titles of the theses are given): T. LALESCU (Bucarest), "Sur l'équation de Volterra"; J. B. ROUSIER (Vienna), "Ondes par émersion"; C. POPOVICI (Jasey, Roumania), "Sur les surfaces intégrales communes aux équations différentielles"; N. ENACHE (Dozesci, Roumania), "Contributions à la théorie de l'écoulement sur les déversions à minces parois et à nappe noyée au dessous"; H. B. HEYWOOD (Peklam, England), "Sur l'équation fonctionnelle de Fredholm et quelques-unes de ses applications."

AT the Vienna meeting of the Astronomische Gesellschaft it was voted to solicit funds for the erection of a Gauss tower on Hohenhagen, the highest mountain near Göttingen, at the vertex of the Brocken-Göttingen-Hohenhagen triangle with which Gauss experimented concerning non-euclidean geometry. Ten thousand Marks have already been received. It is desired that this sum may be sufficiently increased to warrant the laying the corner stone April 30, 1909, the birthday of Gauss. Contributions may be sent to Professor F. Klein, Göttingen, Wilhelm Weberstrasse 3.

THE reproduction of an unpublished photograph of DIRICHLET, superior to any heretofore obtainable, will be issued, provided enough subscriptions are received to defray the expense. The size will be 14 × 18 cm. and the price is two Marks. Orders may be sent to Frl. Lotte Nelson, Darmstadt, Moosbergstrasse 43.

DR. F. RUSL, docent in mathematics at the Bohemian University of Prague, has been appointed associate professor of mathematics at the Bohemian technical school of Prague.

PROFESSOR R. RAU, of the University of Jena, will retire from his university position at the close of the present semester.

PROFESSOR M. DELASSUS, of the University of Besançon, has accepted the professorship of mechanics at the University of Bordeaux.

DR. G. D. BIRKHOFF, of the University of Wisconsin, has accepted an assistant professorship of mathematics at Princeton University.

PROFESSOR OTTO RUPP, of the technical school at Brünn, died in January, at the age of 54 years.

MR. MARTIN SCHILLING, manufacturer of mathematical models and apparatus at Halle, died December 24, 1908, at the age of 42 years.

PROFESSOR E. A. LEGOUX, of the University of Toulouse, died January 6, at the age of 67 years.

PROFESSOR HERMANN MINKOWSKI, of the University of Göttingen, died January 12. He was born at Alexoten, Russia, June 22, 1864, studied at Königsberg and Berlin, held associate professorships of mathematics at Bonn and Königsberg, and was promoted to a full professorship at Königsberg in 1895. The next year he was called to the technical school at Zürich; he accepted the newly established professorship of mathematics at Göttingen in 1903.

RECENT catalogues of second-hand mathematical books: — Bowes and Bowes, 1 Trinity Street, Cambridge, England, catalogue No. 326, later period, about 2600 titles. — A. Hermann, 6 rue de la Sorbonne, Paris, catalogue No. 95, 1900 titles.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BECKER (J.). Die Riccatische Differentialgleichung. (Progr.) Karlsbad, 1908. 8vo. 25 pp.

BÖRNER (E.). Beweis eines Satzes aus der Gruppentheorie. (Progr.) Wien, 1908. 8vo. 13 pp.

BÜCHER, neue, über Naturwissenschaften und Mathematik. Mitgeteilt Herbst 1908. Leipzig, Hinrich, 8vo. 19 pp. M. 0.30

CZUBER (W.). Einführung in die höhere Mathematik. Leipzig, Teubner, 1909. 8vo. 10 + 382 pp. Cloth. M. 12.00.

DIECK (W.). Zur Klassifikation der Punktepaar- und Kegelschnitt-Büschel. (Progr.) Sterkrade, 1908. 8vo. 40 pp.

DINTZL (E.). Einführung in die Funktionenlehre für Schüler der oberen Klassen an Mittelschulen. (Progr.) Wien, 1908. 8vo. 49 pp.

GARCÍA ANTÚNEZ (C.). Complemento del cálculo infinitesimal. Procedimientos geométricos y aparatos de integración. Guadalajara, Ramírez, 1908. 120 pp. P. 5.00

GEIGER (K.). Dreiteilung eines Winkels a) mit Hilfe einer Kurve dritter Ordnung, b) mittels einer Hyperbel, deren Asymptotenwinkel 120° beträgt, und c) der geometrische Beweis der Unmöglichkeit einer elementaren Lösung. (Progr.) Landshut, 1908. 8vo. 25 pp.

GROHMANN (A.). Die verschiedenen Zahlensysteme und Umwandlung von Zahlen in verschiedene Systeme. (Progr.) Wien, 1907. 8vo. 14 pp.

HALLER (S.). Ueber die Brennpunktskurven von Kegelschnittsystemen. (Progr.) München, 1908. 8vo. 35 pp.

HARNACK (A.). See SERRET (J. A.).

HESS (W.). Beweis des grossen Fermatschen Satzes für Ungerades $n > 1$. Dresden, Köhler, 1908. 8vo. 4 pp. M. 0.50

——. Weiteres über den grossen Fermatschen Satz. Dresden, Köhler, 1908. 8vo. 4 pp. M. 0.50

HINRICHS (J.). Aufgaben aus der analytischen Geometrie für die Prima. (Progr.) Neustrelitz, 1908. 8vo. 55 pp.

KLOBASA (C.). Ueber Pythagoräische und Heronische Zahlen. (Progr.) Troppau, 1908. 8vo. 20 pp.

KLUG (R.). Des kaiserlichen Mathematikers Johannes Kepler Neujahrsgeschenk oder über die Sechseckform des Schnees. 1611. Aus dem Lateinischen übersetzt und mit Anmerkungen versehen. (Progr.) Linz, 1907. 8vo. 30 pp.

KOWALEWSKI (G.). Grundzüge der Differential- und Integralrechnung. Leipzig, Teubner, 1909. 8vo. 6 + 452 pp. Cloth. M. 12.00

MORGENSTERN (A.). Beiträge zur numerischen Lösung der Gleichungen fünften Grades. (Progr.) Berlin, 1908. 8vo. 47 pp.

PASCH (M.). Grundlagen der Analysis. Ausgearbeitet unter Mitwirkung von C. Thaer. Leipzig, Teubner, 1909. 8vo. 140 pp. M. 4.00

RENNER (J.). Die Theorie der Abbildung. (Progr.) Graz, 1908. 8vo. 50 pp.

RÖLLNER (F.). Ueber vereinfachte Grundlegung und Systematik der Geometrie. (Progr.) Iglau, 1908. 8vo. 11 pp.

SAXELBY (F. M.). An introduction to practical mathematics. New York, Longmans, 1908. 12mo. 6 + 220 pp. Cloth. $0.80

SCHEFFERS (G.) See SERRET (J. A.).

SERRET (J. A.). Lehrbuch der Differential- und Integralrechnung. Nach Axel Harnack's Uebersetzung, 4te und 5te Auflage, bearbeitet von G. Scheffers. Vol. I: Differentialrechnung. Leipzig, Teubner, 190 . 8vo. 16 + 626 pp. Cloth. M. 13.00

THAER (C.). See PASCH (M.).

WEIGELIN (G.). Der Fermat-Satz und sein Beweis. 2ter Teil. Stuttgart, Enderlen, 1908. 8vo. 16 pp. M. 1.50

WEINFELD (K.). Zur Konstruktion des oskulierenden Hyperboloides an windschiefe Regelflächen. (Progr.) Jägerndorf, 1908. 8vo. 16 pp.

WIELEITNER (H.). Spezielle ebene Kurven. (Sammlung Schubert, LVI.) Leipzig, Göschen, 1908. 8vo. 16 + 409 pp. Cloth. M. 12.00

WINKLER (J.). Die Schwingungsfiguren in analytischer Behandlung. (Progr.) Cilli, 1908. 8vo. 32 pp.

II. ELEMENTARY MATHEMATICS.

BAUDOIN (P.). See BOURLET (C.).

BERTRAND (J.) et GARCET (H.). Traité d'algèbre. 1re partie à l'usage des classes de mathématiques élémentaires; 2e partie à l'usage des classes de mathématiques spéciales. Nouvelle édition. Paris, Hachette, 1908. 8vo. 2 vols. 1re partie, 4 + 326 pp.; 2e partie, 392 pp. Fr. 10.00

BOREL (E.) et ROYER (M.). Algèbre à l'usage des écoles normales, primaires (programmes de 1905) des écoles primaires supérieures, des aspirants et aspirantes aux brevets de l'enseignement primaire et des lycées et collèges de jeunes filles. Paris, Colin, 1908. 16mo. 8 + 469 pp. Fr. 3.50

BOURLET (C.). Cours abrégé de géométrie dans l'espace, rédigé conformément aux programmes de 1905. Corrigés des 344 exercices et problèmes; avec la collaboration de P. Baudoin. Paris, Hachette, 1908. 16mo. 176 pp. Fr. 2.00

BOUVART (C.) et RATINET (A.). Nouvelles tables de logarithmes à cinq décimales. Table numérique. Table trigonométrique. Division centésimale établie conformément à l'arrêté ministériel du 3 août 1901, à l'usage des candidats au baccalauréat et aux écoles polytechnique et de Saint-Cyr. 5e édition. Paris, Hachette, 1908. 8vo. 128 pp. Fr. 2.00

COQUELET (G.). Mathématiques spéciales. Concours d'admission à l'école polytechnique, à l'école centrale des arts et manufactures, à l'école des ponts et chaussées, etc. Préface de J. J. Pillet. Edition complète. Paris, Delagrave, 1909. 8vo. 215 pp. Fr. 5.00

CRACKNELL (A. G.). See WORKMAN (W. P.).

DRUXES (J.). See HEIS (E.).

FRONTERA (D. J.). See SONNET (H.).

GARCET (H.). See BERTRAND (J.).

GÓMEZ PALLETE (J.). Trigonometría, rectilinea y esférica, considerada como introducción al estudio de la geometría analítica. 11a edición aumentada con una escogida colección de ejercicios mineros. Madrid, Perlado, 1908. 144 pp. P. 5.00

HEIS (E.). Sammlung von Beispielen und Aufgaben aus der allgemeinen Arithmetik und Algebra zum Gebrauche an höheren Schulen. 112te Auflage. Nach den zeitgemässen Anforderungen des mathematischen Unterrichts neubearbeitet und erweitert von J. Druxes. 2ter Teil. Pensum der Ober-Sekunda, Unter- und Ober-Prima. Köln, Du Mont-Schauberg, 1908. 8vo. 7 + 273 pp. M. 2.80

LIEBER (H.). See MÜSEBECK (C.).

MÜSEBECK (C.). Aufgaben für den Unterricht in der Stereometrie und sphärischen Trigonometrie, enthaltend Aufgaben zu Teil III des Leitfadens der Elementar-Mathematik von H. Lieber und F. von Lühmann. Berlin, Simion, 1908. 8vo. 6 + 106 pp. M. 1.80

PILLET (J. J.). See COQUELET (G.).

RATINET (A.). See BOUVART (C.).

ROYER (M.). See BOREL (E.).

SCHMEHL (C.). Sammlung von Aufgaben aus der Algebra und algebraischen Analysis. Für die Prima der realistischen Anstalten. Auflösungen. Giessen, Roth, 1909. 8vo. 69 pp. M. 1.80

SCHUBERT (H.). Beispiel-Sammlung zur Arithmetik und Algebra. 3te, durchgesehene Auflage. Neudruck. (Sammlung Göschen, 48.) Leipzig, Göschen, 1908. 16mo. 147 pp. Cloth. M. 0.80

SMITH (D. E.). Rara arithmetica. Boston, Ginn, 1908. 8vo. xvi + 506 pp. Edition de luxe. 2 vols. Vellum. $20.00

SONNET (H.). Primeros elementos de geometría con los principales aplicaciones al dibujo lineal, al levantamiento de planos, á la agrimensura, etc., etc. Traducidos del francés par D. J. Frontera. 8a edición. Paris, Hachette, 1908. 16mo. 239 pp. Fr. 3.00

WORKMAN (W. P.) and CRACKNELL (A. G.). Intermediate geometry. Being sections 5 and 6 of geometry, theoretical and practical. London, Clive, 1908. 8vo. 204 pp. 2s. 6d.

——. Geometry, theoretical and practical. Part III. London, Clive, 1908. 8vo. 76 pp. 1s. 6d.

III. APPLIED MATHEMATICS.

BEZINE (E.). Eléments de cinématique théorique et appliquée. Paris, Loubat, 1908. 8vo. 573 pp.

CASSELL'S cyclopedia of mechanics. 5 vols. London, Cassell, 1908. 8vo. Cloth. Each, 7s. 6d.

DYNAMICS, NOTES ON. 2d edition. London, Wyman, 1908. 3 s.

FUZET (H.) et RECLUS (L.). Précis de mathématiques commerciales et financières. A l'usage des élèves des écoles supérieures du commerce, des candidats à la Banque de France, au professorat commercial et au professorat de comptabilité. Paris, Delagrave, 1908. 18mo. 6 + 517 pp. Fr. 5.50

GIROD (J.). Précis de géométrie descriptive et de géométrie cotée, conforme aux programmes de 1905. 1re partie. Pour les classes de 1re C et D. Paris, Alcan, 1908. 8vo. 2 + 204 pp. Fr. 2.50

——. Précis de géométrie descriptive et de géométrie cotée, conforme aux programmes de 1905. 2e partie. Pour les classes de mathématiques A et B. Paris, Alcan, 1909. 8vo. 272 pp. Fr. 3.50

GÜNTHER (L.). Die Mechanik des Weltalls. Eine volkstümliche Darstellung der Lebensarbeit Johannes Keplers, besonders seiner Gesetze und Probleme. Leipzig, Teubner, 1909. 8vo. 16 + 156 pp. M. 2 50

HAUSSNER (R.). Darstellende Geometrie. 2ter Teil. Perspektive ebener Gebilde; Kegelschnitte. (Sammlung Göschen, 143.) Leipzig, Göschen, 1908. 16mo. 164 pp. Cloth. M. 0.80

IKLÉ (M.). See POINCARÉ (H.).

POINCARÉ (H.). Die Maxwellsche Theorie und die Hertzschen Schwingungen. Die Telegraphie ohne Draht. Aus dem Französischen von M. Iklé. Leipzig, Barth, 1909. 8vo. 199 pp. M. 3.20

——. Leçons de mécanique céleste professées à la Sorbonne. Vol. 2. 2e partie: Théorie de la lune. Paris, Gauthier-Villars, 1909. 8vo. 142 pp. Fr. 5.00

RECLUS (L.). See FUZET (H.).

SCHWAIGER (A.). Das Regulierproblem in der Elektrotechnik. Leipzig, Teubner, 1909. 8vo. 5 + 102 pp. Cloth. M. 3.60